Math 3280 Tutorial 1

Basic counting models.

Model 1. (The basic principle of counting). Let E be an experiment which consists of k independent experiments $E_1, E_2, \cdots, E_k$. Suppose that E_i has n_i outcomes $(i=1,\cdots,k)$, then there are

$$n_1 n_2 \cdots n_k$$

possible outcomes of the Experiment E.

Proof: Each outcome of E can be represented by a vector

$$(a_1, a_2, \cdots a_k)$$

where a_i denotes an outcome of E_i $(i=1,\cdots,k)$. Conversely, each such vector gives an outcome of E. Since there are

$$n_1 n_2 \cdots n_k$$

such vectors, there are $n_1 n_2 \cdots n_k$ outcomes of E.

Example 1: Two dice are rolled. Denote the set of outcomes by $\{(i,j) : 1 \leq i, j \leq 6,\ i,j \in \mathbb{N}\}$. There are $\underset{\shortparallel\ 6\times 6}{\underline{36}}$ possible outcomes.

Model 2. (Permutation). There are $n!$ $(n! = 1\times 2\times \cdots \times n)$ different ordered arrangements of n disjoint objects.

Example 2: There are $n!$ possible ways to arrange n persons in a queue.

Model 3. Suppose in an experiment, one needs to select k distinct objects from a set of n objects. If the order of selection is considered, then there are

$$\boxed{\frac{n!}{(n-k)!}}$$

possible ways of selection.

Pf: $\overset{1\cdot 2\cdots k}{n(n-1)\cdots(n-k+1)} = \frac{n!}{(n-k)!}$

Example 3: Suppose one needs to select k persons from n persons to take k distinct jobs, there are $\frac{n!}{(n-k)!}$ possible ways.

Model 4. In model 3, if the order of selection is not considered, there are

$$\binom{n}{k} := \frac{n!}{(n-k)!\,(k!)}$$

possible ways of selection.

Proof: Since all the $k!$ permutations of each outcome in model 3 correspond the same outcome in model 4.

$(1,2,\cdots,k)$ different in model 3, the same outcome in model 4.
$(2,1,3,\cdots,k)$ $k!$ permutations.

Example 4: There are $\binom{n}{k}$ possible ways to select k persons from n persons to form a group.

Model 5. Suppose n objects are to be devided into k

groups of respective size $n_1, n_2, \dots, n_k$, where $\underline{n_1+n_2+\dots+n_k=n}$.
If these k groups are ordered, there are

$$\boxed{\binom{n}{n_1, n_2, \dots, n_k}} = \frac{n!}{n_1! \cdot n_2! \cdots n_k!}$$

possible divisions.

Proof: Since k-groups are ordered,

$$\begin{matrix} 1 & 2 & 3 & \cdots & k \\ \binom{n}{n_1} \cdot & \binom{n-n_1}{n_2} & \binom{n-(n_1+n_2)}{n_3} & \cdots & \binom{n-(n_1+n_2+\dots+n_{k-1})}{n_k} \end{matrix}$$

$$= \frac{n!}{n_1! \cdot \cancel{(n-n_1)!}} \cdot \frac{\cancel{(n-n_1)!}}{n_2! \cdot \cancel{(n-(n_1+n_2))!}} \cdot \frac{\cancel{(n-(n_1+n_2))!}}{n_3! \cdot (n-(n_1+n_2+n_3))!} \cdots \frac{(n-(n_1+n_2+\dots+n_{k-1}))}{n_k! \cdot \underbrace{(n-(n_1+\dots+n_k))!}_{\substack{\| \\ 0!=1}}}$$

$$= \frac{n!}{n_1! \cdot n_2! \cdots n_k!}$$

Example 5: In a game of bridge, 52 cards are distributed to 4 players. There are

$$\binom{52}{13, 13, 13, 13} = \frac{52!}{(13!) \cdot (13!) \cdot (13!) (13!)}$$

possible divisions.

Model 6. In model 5, if the k groups are ordered, then there are

$$\binom{n}{n_1, n_2, \dots, n_k} \cdot \frac{1}{\boxed{k!}} = \frac{n!}{n_1! \cdot n_2! \cdots n_k! \cdot k!}$$

possible ways of division.

Example 6: Suppose 52 cards on the table, and one is asked to divided them into 4 piles such that each pile consists of 13 cards. There are

$$\binom{52}{13,13,13,13} \cdot \frac{1}{4!}$$

possible results.

Ex 1: A pair of die is rolled. what is the probability that the second die lands on a larger value of the first?

Solution: Denote by (i,j) the outcome of the first die gives i and the second gives j.

Sample space $= \{(i,j) : i,j \in \{1,2,\cdots,6\}\}$. $6 \times 6 = 36$ possible outcomes.

E_1 denotes the event that the second is larger than the first one,

E_2 denotes - - - - - - first - - second one.

E_3 - - the second is equal to the first.

$$P(E_1) = P(E_2)$$

$$P(E_1) + P(E_2) + P(E_3) = 1.$$

$$P(E_3) = \frac{6}{36}$$

$$P(E_1) = \frac{1 - P(E_3)}{2} = \frac{5}{12}.$$